The predictability problem in systems with an uncertainty in the evolution law

2000 J. Phys. A: Math. Gen. 33 1313

(http://iopscience.iop.org/0305-4470/33/7/302)

J. Phys. A: Math. Gen. **33** (2000) 1313–1324. Printed in the UK PII: S0305-4470(00)05594-3

The predictability problem in systems with an uncertainty in the evolution law

G Boffetta†, A Celani†, M Cencini‡, G Lacorata§ and A Vulpiani‡

† Dipartimento di Fisica Generale, Università di Torino, Via Pietro Giuria 1, 10125 Torino, Italy
Istituto Nazionale Fisica della Materia, Unità di Torino Università, Istituto di Cosmogeofisica del CNR, C. Fiume 4, 10133 Torino, Italy
‡ Dipartimento di Fisica, Università di Roma 'la Sapienza', Piazzale Aldo Moro 5, 00185 Roma, Italy
Istituto Nazionale Fisica della Materia, Unità di Roma 1
§ Dipartimento di Fisica, Università dell' Aquila, Via Vetoio 1, 67010 Coppito, L'Aquila, Italy
Istituto di Fisica dell'Atmosfera, CNR, Via Fosso del Cavaliere, 00133 Roma, Italy

Received 30 June 1999

Abstract. The problem of unpredictability in a physical system due to the incomplete knowledge of the evolution laws is addressed. Major interest is devoted to the analysis of error amplification in chaotic systems with many characteristic times and scales when the fastest scales are not resolved. The parametrization of the unresolved scales introduces a non-infinitesimal uncertainty (with respect to the true evolution laws) which affects the forecasting ability on the large resolved scales. The evolution of non-infinitesimal errors from the unresolved scales up to the large scales is analysed by means of the finite-size Lyapunov exponent. It is shown that proper parametrization of the unresolved scales allows one to recover the maximal predictability of the system.

1. Introduction

The ability to predict the future state of a system, given its present state, stands at the foundations of scientific knowledge with relevant implications from an applicative point of view in geophysical and astronomical sciences. In the prediction of the evolution of a physical system, e.g. the atmosphere, we are severely limited by the fact that we do not know the evolution equations and the initial conditions with arbitrary accuracy. Indeed, one integrates a mathematical model given by a finite number of equations. The initial condition, a point in the phase space of the model, is only determined with a finite resolution (i.e., by a finite number of observations) [1].

Using the concepts of dynamical systems theory, there has been some progress made in understanding the growth of an uncertainty during the time evolution. An infinitesimal initial uncertainty ($\delta_0 \to 0$) in the long-time limit ($t \to \infty$) grows exponentially in time with a typical rate given by the leading Lyapunov exponent λ, $|\delta x(t)| \sim \delta_0 \exp(\lambda t)$. Therefore, if our purpose is to forecast the system within a tolerance Δ, the future state of the system can only be predicted up to the *predictability time*, given by

$$T_p \simeq \frac{1}{\lambda} \ln\left(\frac{\Delta}{\delta_0}\right). \tag{1}$$

In the literature, the problem of predictability with respect to uncertainty on the initial conditions is referred to as *predictability of the first kind*.

In addition, in real systems we must also cope with the lack of knowledge of the evolution equations. Let us consider a system described by a differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(t) = \boldsymbol{f}(\boldsymbol{x}, t) \qquad \boldsymbol{x f} \in \mathcal{R}^n. \tag{2}$$

As a matter of fact, we do not know the equations exactly, and we have to devise a model which is different from the true dynamics:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(t) = \boldsymbol{f}_\epsilon(\boldsymbol{x}, t) \qquad \text{where} \quad \boldsymbol{f}_\epsilon(\boldsymbol{x}, t) = \boldsymbol{f}(\boldsymbol{x}, t) + \epsilon\delta\boldsymbol{f}(\boldsymbol{x}, t). \tag{3}$$

Therefore, it is natural to wonder about the relation between the true evolution (*reference* or *true* trajectory $\boldsymbol{x}_T(t)$) given by (2) and the one effectively computed (*perturbed* or *model* trajectory $\boldsymbol{x}_M(t)$) given by (3). This problem is referred to as *predictability of the second kind*.

Let us make some general remarks. At the foundation of the problem of predictability of the second kind exists the issue of *structural stability* [2]: since the evolution laws are only known with finite precision it is highly desirable that at least certain properties are not too sensitive to the details of the equations of motion. For example, in a system with a strange attractor, small generic changes in the evolution laws should not drastically change the dynamics (see [3, 4] for a discussion of non-generic perturbations).

In chaotic systems the effects of a small generic uncertainty on the evolution law are similar to those due to the finite precision on the initial condition [5]. The model trajectory of the perturbed dynamics diverges exponentially from the reference one with a mean rate given by the Lyapunov exponent of the original system.

In dynamical systems theory, the first- and second-kind predictability problems are understood essentially in the limit of infinitesimal perturbations. However, even in this limit we must also consider the fluctuations of the rate of expansion which can lead to relevant modifications of the predictability time (1), in particular for strongly intermittent systems [6–8].

As far as finite perturbations are considered, the leading Lyapunov exponent is not relevant for the predictability issue. In the presence of many characteristic times and spatial scales the Lyapunov exponent is related to the growth of small-scale perturbations which saturates for short times and has very little relevance for the growth of large-scale perturbations [1, 9, 10]. To overcome this shortcoming, a suitable characterization of the growth of non-infinitesimal perturbations, in terms of the finite-size Lyapunov exponent (FSLE), has been recently introduced [11, 12].

Additionally, in the case of predictability of the second kind, one often has to deal with errors which are far from infinitesimal. Indeed, in real systems the size of an uncertainty on the evolution equations is determinable only *a posteriori*, based on the ability of the model equations to reproduce some of the features of the phenomenon. Typical examples are systems described by partial differential equations (e.g. turbulence, atmospheric flows). The study of these systems is performed by using a numerical model with unavoidable severe approximations, the most relevant of which is the necessity to cut off some degrees of freedom (i.e. the small-scale variables).

The aim of this paper is to analyse the effects of limited resolution to our ability to forecast the large-scale features. This raises two problems: in the first place one has to deal with perturbations of the evolution equations which, in general, cannot be considered small; second, the parametrization of the unresolved modes. The FSLE fits the first point very well, being able to characterize and quantify the effects of uncertainty on the evolution laws at different scales. As regards the second point, one can define the optimal parametrization as that one for which the predictability on the large scales is no worse than the intrinsic predictability of the system (i.e. due to uncertainties on the initial conditions). We shall show that with some phenomenological parametrization of the small scales one can recover on large scales the intrinsic predictability of the system. This is far from clear in systems with some feedback from the large scales toward the small ones.

This paper is organized as follows. In section 2 we report some known results concerning the predictability problem of the second kind and recall the definition of the FSLE. In section 3 we present numerical results on a simple model. In section 4 we consider more complex systems with many characteristic times. In section 5 we present some conclusions. In the appendix we discuss the problem of the parametrization of the unresolved variables.

2. Effects of a small uncertainty on the evolution law

In the second-kind predictability problem, we can distinguish three general cases depending on the original dynamics. In particular, equation (2) may display:

(i) trivial attractors: asymptotically stable fixed points or attracting periodic orbits;
(ii) marginally stable fixed points or periodic/quasi-periodic orbits as in integrable Hamiltonian systems;
(iii) chaotic behaviour.

In case (i) small changes in the equations of motion do not modify the qualitative features of the dynamics. Case (ii) is not generic and the outcome strongly depends on the specific perturbation $\delta \boldsymbol{f}$, i.e. it is not structurally stable. In the chaotic case (iii) one expects that the perturbed dynamics is still chaotic. In this paper we will only consider the latter case.

Let us also mention that, in numerical computations of evolution equations (e.g. differential equations), there are two unavoidable sources of error: the finite precision representation of numbers which causes the computer phase space to be necessarily discrete, and the round-off which introduces a sort of noise. As a consequence, any numerical trajectory is periodic. Nevertheless, the period is usually very large, apart from very low computer precision [5]. Here we do not consider this source of difficulties. The round-off produces a perturbation in (2) which can be written as $\delta \boldsymbol{f}(\boldsymbol{x}, t) = \boldsymbol{w}(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x}, t)$ and $\epsilon \sim 10^{-\alpha}$ ($\alpha =$ number of digits in the floating-point representation), where $\boldsymbol{w} = \mathrm{O}(1)$ is an unknown function which may depend on $\boldsymbol{f}$ and on the software [13]. In general, the round-off error is very small and may have a positive role in selecting the physical probability measure, the so-called *natural measure*, from the set of ergodic invariant measures [14].

In chaotic systems the effects of a small uncertainty on the evolution law is, from many aspects, similar to those due to imperfect knowledge of the initial conditions. This can be understood through the following example. Consider the Lorenz system [15]

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= \sigma(y - x) \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= Rx - y - xz \\ \frac{\mathrm{d}z}{\mathrm{d}t} &= xy - bz. \end{aligned} \tag{4}$$

In order to mimic an experimental error in the determination of the evolution law we consider a small error ϵ on the parameter R: $R \to R + \epsilon$. Let us consider the difference $\Delta\boldsymbol{x}(t) = \boldsymbol{x}_M(t) - \boldsymbol{x}_T(t)$ with, for simplicity, $\Delta\boldsymbol{x}(0) = 0$, i.e. we assume a perfect knowledge of the initial conditions. One has, with obvious notation

$$\frac{\mathrm{d}\Delta\,\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{f}_\epsilon(\boldsymbol{x}_M) - \boldsymbol{f}(\boldsymbol{x}_T) \simeq \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\Delta\boldsymbol{x} + \frac{\partial \boldsymbol{f}_\epsilon}{\partial R}\epsilon. \tag{5}$$

At time $t = 0$ one has $|\Delta \boldsymbol{x}(0)| = 0$, therefore $|\Delta \boldsymbol{x}(t)|$ only grows due to the effect of the second term in (5). At later times, when $|\Delta \boldsymbol{x}(t)| \approx O(\epsilon)$ the first term of (5) becomes the leading one, and we recover predictability of the first kind for an initial uncertainty $\delta_0 \sim \epsilon$. Therefore, apart from an initial growth, which depends on the specific perturbation, the evolution of $\langle \log(|\Delta \boldsymbol{x}(t)|\rangle$ follows the usual linear growth with the slope given by the leading Lyapunov exponent. Typically, the value of the Lyapunov exponent computed using the model dynamics differs from the true one by a small amount of order ϵ, i.e. $\lambda_M = \lambda_T + O(\epsilon)$ [5].

These considerations only apply to infinitesimal perturbations. The generalization to finite perturbations can be obtained by considering the extension of the Lyapunov exponent to finite errors. The definition of the FSLE $\lambda(\delta)$ is given in terms of the time, $T_r(\delta)$, which a perturbation of initial size δ takes to amplify by a factor r (>1):

$$\lambda(\delta) = \left\langle \frac{1}{T_r(\delta)} \right\rangle_t \ln r \tag{6}$$

where $\langle \cdots \rangle_t$ denotes average with respect to the natural measure, i.e. along the trajectory (for details see [11, 12]). For chaotic systems, in the limit of infinitesimal perturbations ($\delta \to 0$) $\lambda(\delta)$ is simply the leading Lyapunov exponent λ [16]. Let us note that the above definition of $\lambda(\delta)$ is not appropriate to discriminate cases with $\lambda = 0$ and $\lambda < 0$, since the predictability time is positive by definition. Nevertheless, this is not a limitation as long as we deal with chaotic systems.

In many realistic situations the error growth for infinitesimal perturbations is dominated by the fastest scales, which are typically the smallest ones (e.g. small-scale turbulence). When δ is no longer infinitesimal, $\lambda(\delta)$ is given by the fully nonlinear evolution of the perturbation. In general, $\lambda(\delta) \leqslant \lambda$, according to the intuitive picture that large scales are more predictable. Outside the range of scales in which the δ error can be considered infinitesimal, the function $\lambda(\delta)$ depends on the details of the dynamics and in principle on the norm used. In fully developed turbulence one has the universal law $\lambda(\delta) \sim \delta^{-2}$ in the inertial range [11, 12]. In general, the behaviour of $\lambda(\delta)$ as a function of δ gives important information on the characteristic times and scales of the system and it has also been applied to passive transport in closed basins [17].

Let us now return to example (4). We compute $\lambda_{TT}(\delta)$, the FSLE for the true equations, and $\lambda_{TM}(\delta)$, the FSLE computed following the distance between one true trajectory and one model trajectory starting at the same point. These are shown in figure 1. $\lambda_{TT}(\delta)$ displays a plateau indicating a chaotic dynamics with leading Lyapunov exponent $\lambda \simeq 1$. With respect to predictability of the second kind, for $\delta > \epsilon$ the second term in (5) becomes negligible and we observe the transition to the Lyapunov exponent $\lambda_{TM}(\delta) \simeq \lambda_{TT}(\delta) \simeq \lambda$. In this range of errors the model system recovers the intrinsic predictability of the true system. For very small errors, $\delta < \epsilon$, where the growth of the error is dominated by the second term in (5), we have $\lambda_{TM}(\delta) > \lambda_{TT}(\delta)$.

This example shows that it is possible to recover the intrinsic predictability of a chaotic system even in the presence of some uncertainty in the model equations. This is rather clear and intuitive for infinitesimal perturbations. The case of non-infinitesimal perturbation is more subtle.

In the following we will consider more complex situations, in which strongly interacting degrees of freedom with different characteristic times are involved. In these cases the correct parametrization of the unresolved modes is crucial for the prediction of large-scale behaviour.

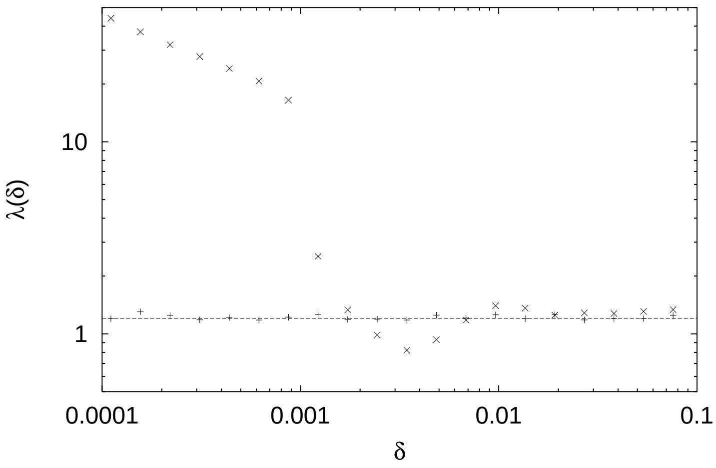

Figure 1. FSLEs $\lambda_{TT}(\delta)$ (+) and $\lambda_{TM}(\delta)$ ($\times$) versus δ for the Lorenz model (4) with $\sigma = c = 10$, $b = \frac{8}{3}$, $R = 45$ and $\epsilon = 0.001$. The dashed line represents the leading Lyapunov exponent for the unperturbed system ($\lambda \approx 1.2$). The statistics is over 10^4 realizations.

3. Systems with two timescales

Before analysing in detail the effects of non-infinitesimal perturbations of the evolution laws in some specific models let us clarify our aims. We consider a dynamical system written in the following form:

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) \qquad \frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t} = \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}) \tag{7}$$

where $\boldsymbol{f}, \boldsymbol{x} \in \mathcal{R}^n$ and $\boldsymbol{g}, \boldsymbol{y} \in \mathcal{R}^m$ and, in general, $n \neq m$. Now, let us suppose that the fast variables $\boldsymbol{y}$ cannot be resolved: a typical example is the sub-grid modes in partial differential equation discretizations. In this framework, a natural question is: how must we parametrize the unresolved modes ($\boldsymbol{y}$) in order to predict the resolved modes ($\boldsymbol{x}$)?

As discussed by Lorenz [10], to reproduce—at a qualitative level—a given phenomenology, e.g. the El Niño southern oscillation (ENSO) phenomenon, one can drop out the small-scale features without negative consequences. However, one unavoidably fails in forecasting the ENSO (i.e., the actual trajectory) without taking into account in a suitable way the small-scale contributions. Indeed it is not obvious that by introducing a finite perturbation one can recover the intrinsic predictability of the system on large scales.

An example in which it is relatively simple to develop a model for the fast modes is represented by the skew systems:

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) \qquad \frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t} = \boldsymbol{g}(\boldsymbol{y}). \tag{8}$$

In this case, the fast modes ($\boldsymbol{y}$) do not depend on the slow ones ($\boldsymbol{x}$). One can expect that in this case, neglecting the fast variables or parametrizing them with a suitable stochastic process, should not drastically affect the prediction of the slow variables [18].

On the other hand, if $\boldsymbol{y}$ feels some feedback from $\boldsymbol{x}$, we cannot simply neglect the unresolved modes. In the appendix we discuss this point in detail. In practice one has to construct an effective equation for the resolved variables:

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{f}_M(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x})) \tag{9}$$

where the functional form of $\boldsymbol{y}(\boldsymbol{x})$ and $\boldsymbol{f}_M$ are found by phenomenological arguments and/or by numerical studies of the full dynamics.

Let us now investigate an example with a recently introduced toy model of atmospheric circulation [10,19] including large scales x_k (synoptic scales) and small scales $y_{j,k}$ (convective scales):

$$\begin{aligned} \frac{\mathrm{d}x_k}{\mathrm{d}t} &= -x_{k-1}(x_{k-2} - x_{k+1}) - \nu x_k + F - \sum_{j=1}^{J} y_{j,k} \\ \frac{\mathrm{d}y_{j,k}}{\mathrm{d}t} &= -cby_{j+1,k}(y_{j+2,k} - y_{j-1,k}) - c\nu y_{j,k} + x_k \end{aligned} \tag{10}$$

where $k = 1, \ldots, K$ and $j = 1, \ldots, J$. As in [10] we assume periodic boundary conditions on k ($x_{K+k} = x_k$, $y_{j,K+k} = y_{j,k}$) while for j we impose $y_{J+j,k} = y_{j,k+1}$. The variables x_k represent some large-scale atmospheric quantities in K sectors extending on a latitude circle, while the $y_{j,k}$ represent quantities on smaller scales in $J \cdot K$ sectors. The parameter c is the ratio between the fast and slow characteristic times, and b measures the relative amplitude.

As pointed out by Lorenz, this model shares some basic properties with more realistic models of the atmosphere. In particular, the nonlinear terms, which model the advection, are quadratic and conserve the total kinetic energy $\sum_k (x_k^2 + \sum_j y_{j,k}^2)$ in the unforced ($F = 0$), inviscid ($\nu = 0$) limit; the linear terms containing ν mimic dissipation and the constant term F acts as an external forcing preventing the total energy from decaying.

If one is interested in forecasting the large-scale behaviour of the atmosphere by using only the slow variables, a natural choice for the model equations is

$$\frac{\mathrm{d}x_k}{\mathrm{d}t} = -x_{k-1}(x_{k-2} - x_{k+1}) - \nu x_k + F - G_k(\boldsymbol{x}) \tag{11}$$

where $G_k(\boldsymbol{x})$ represents the parametrization of the fast components in (10) (see the appendix).

The FSLE for the true system [20], $\lambda_{TT}(\delta)$, is shown in figure 2 and displays the two characteristic plateau corresponding to a fast component (for $\delta \ll 0.1$) and a slow component for large δ. Figure 2 also shows what happens when one simply neglects the fast components $y_{j,k}$ (i.e. $\boldsymbol{G}(\boldsymbol{x}) = 0$). At very small δ one has $\lambda_{TM}(\delta) \simeq \delta^{-1}$. This behaviour can be understood by noting that if $\delta \ll \epsilon$, the error grows as $\mathrm{d}\delta/\mathrm{d}t \sim \epsilon$ and thus $\lambda_{TM} \sim \epsilon/\delta$.

For large errors we observe that, with this rough approximation, we are not able to capture the intrinsic predictability of the original system. More refined parametrizations in terms of stochastic processes with the correct probability distribution function and correlation times do not improve the forecasting ability.

The reason for this failure is due to the presence of a feedback term in the equations (10) which induces strong correlations between the variable x_k and the unresolved coupling $\sum_{j=1}^{J} y_{j,k}$. For a proper parametrization of the unresolved variables we follow the strategy discussed in the appendix. Basically, we adopt

$$G(\boldsymbol{x}) = \nu_e x_k \tag{12}$$

in which ν_e is a numerically determined parameter. Figure 2 shows that, although small scales are not resolved, the intrinsic large-scale predictability is well reproduced and one has $\lambda_{TM}(\delta) \simeq \lambda_{TT}(\delta)$ for large δ.

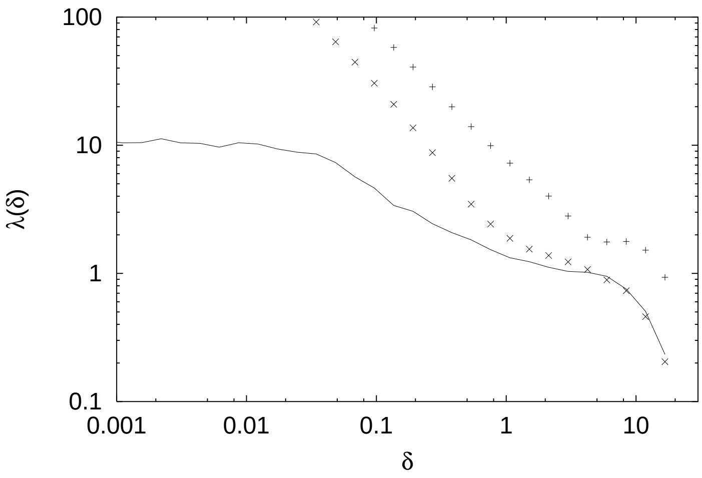

Figure 2. FSLEs for the Lorenz '96 model $\lambda_{TT}(\delta)$ (solid curve) and $\lambda_{TM}(\delta)$ versus δ obtained by dropping the fast modes (+) and with eddy-viscosity parametrization (×) as discussed in (11) and (12). The parameters are $F = 10$, $K = 36$, $J = 10$, $\nu = 1$ and $c = b = 10$, implying that the typical y variable is ten times faster and smaller than the x variable. The value of the parameter $\nu_e = 4$ is chosen after a numerical integration of the complete equations as discussed in the appendix. The statistics is over 10^4 realizations.

4. Large-scale predictability in a turbulence model

We now consider a more complex system which mimics the energy cascade in fully developed turbulence. The model is in the class of the so-called *shell models* introduced some years ago for a dynamical description of small-scale turbulence. For a recent review on shell models see [21]. This model has relatively few degrees of freedom but involves many characteristic scales and times. The velocity field is assumed isotropic, and it is decomposed on a finite set of complex velocity components u_n representing the typical turbulent velocity fluctuation on a 'shell' of scales $\ell_n = 1/k_n$. In order to reach very high Reynolds number with a moderate number of degrees of freedom, the scales are geometrically spaced as $k_n = k_0 2^n$ $(n = 1, \ldots, N)$.

The specific model considered here has the form [22]

$$\frac{\mathrm{d}u_n}{\mathrm{d}t} = \mathrm{i}(k_{n+1}u^*_{n+1}u_{n+2} - \tfrac{1}{2}k_n u^*_{n-1}u_{n+1} + \tfrac{1}{2}k_{n-1}u_{n-2}u_{n-1}) - \nu k_n^2 u_n + f_n \qquad (13)$$

where ν represent the kinematic viscosity and f_n is a forcing term which is only restricted to the first two shells (in order to mimic large-scale energy injection).

Without entering into the details, we recall that the shell model (13) displays an energy cascade *à la* Kolmogorov from large scales (small n) to dissipative scales ($n \sim N$) with a statistical stationary energy flux. Scaling laws for the average velocity components are observed:

$$\langle |u_n^p| \rangle \simeq k_n^{-\zeta_p} \qquad (14)$$

with exponents close to the Kolmogorov 1941 values $\zeta_p = p/3$.

From a dynamical point of view, model (13) displays complex chaotic behaviour which is responsible of the small deviation of the scaling exponents (intermittency) with respect to the Kolmogorov values. Neglecting the (small) intermittency effects, a dimensional estimate of the characteristic time (eddy turnover time, i.e. the typical time after which $u_n(t)$ is practically independent of $u_n(0)$) for scale n gives

$$\tau_n \simeq \frac{\ell_n}{|u_n|} \simeq k_n^{-2/3}. \tag{15}$$

The scaling behaviour holds up to the Kolmogorov scale $\eta = 1/k_d$ defined as the scale at which the dissipative term in (13) becomes relevant. The Lyapunov exponent of the turbulence model can be estimated as the fastest characteristic time τ_d and one has the prediction [14]

$$\lambda \sim \frac{1}{\tau_d} \sim Re^{1/2} \tag{16}$$

where we have introduced the Reynolds number $Re \propto 1/\nu$. It is possible to predict the behaviour of the FSLE by observing that the faster scale k_n at which an error of size δ is still active (i.e. below the saturation) is such that $u_n \simeq \delta$. Thus $\lambda(\delta) \sim 1/\tau_n$ and, using Kolmogorov scaling, one obtains

$$\lambda_{TT}(\delta) \sim \begin{cases} \lambda & \text{for} \quad \delta \leqslant u_d \\ \delta^{-2} & \text{for} \quad u_d \leqslant \delta \leqslant u_0. \end{cases} \tag{17}$$

To be more precise, there is an intermediate range between the two shown in (17). In addition, it is remarkable that the prediction $\lambda_{TT}(\delta) \sim \delta^{-2}$, which can be derived within the multifractal model for turbulence, it is not affected by intermittency and gives the law originally proposed by Lorenz [23]. For a detailed discussion see [11, 12].

In order to simulate a finite resolution in the model, we consider a modelization of (13) in terms of an eddy viscosity [24]

$$\frac{\mathrm{d}u_n}{\mathrm{d}t} = \mathrm{i}(k_{n+1}u_{n+1}^* u_{n+2} - \tfrac{1}{2}k_n u_{n-1}^* u_{n+1} + \tfrac{1}{2}k_{n-1}u_{n-2}u_{n-1}) - \nu_n^{(e)} k_n^2 u_n + f_n \tag{18}$$

where now $n = 1, \ldots, N_M < N$ and the eddy viscosity, restricted to the last two shells, has the form

$$\nu_n^{(e)} = \kappa \frac{|u_n|}{k_n} (\delta_{n,N_M-1} + \delta_{n,N_M}) \tag{19}$$

where κ is a constant of order 1 (see the appendix). The model equations (18) are analogous to the large eddy simulation (LES) in the shell model, which is one of the most popular numerical method for integrating large-scale flows. Thus, although shell models are not realistic models for large-scale geophysical flows (being, nevertheless, a good model for small-scale turbulent fluctuations), the study of the effect of truncation in terms of eddy viscosity is of general interest.

In figure 3 we show $\lambda_{MM}(\delta)$, i.e. the FSLE computed for the model equations (18) with $N = 24$ at different resolutions $N_M = 9, 15, 20$. A plateau is detected for small δ error amplitudes corresponding to the leading Lyapunov exponent, which increases with increasing resolution—being proportional to the fastest timescale—according to $\lambda \sim k_{N_M}^{2/3}$. At larger δ the curves collapse onto the $\lambda_{TT}(\delta)$, showing that the large-scale statistics of the model is not affected by the small-scale resolution.

The capability of the model to satisfactorily predict the features of the 'true' dynamics is not determined by $\lambda_{MM}(\delta)$ anyway, but by $\lambda_{TM}(\delta)$, as shown in figure 4.

Increasing the resolution $N_M = 9, 15, 20$ towards the fully resolved case $N = 24$ the model improves, in agreement with the expectation that λ_{TM} approaches λ_{TT} for a perfect model. At large δ the curves practically coincide, showing that the predictability time for large error sizes (associated with large scales) is independent of the details of small-scale modelling. Better resolved models achieve $\lambda_{TM} \simeq \lambda_{TT}$ for smaller values of the error δ.

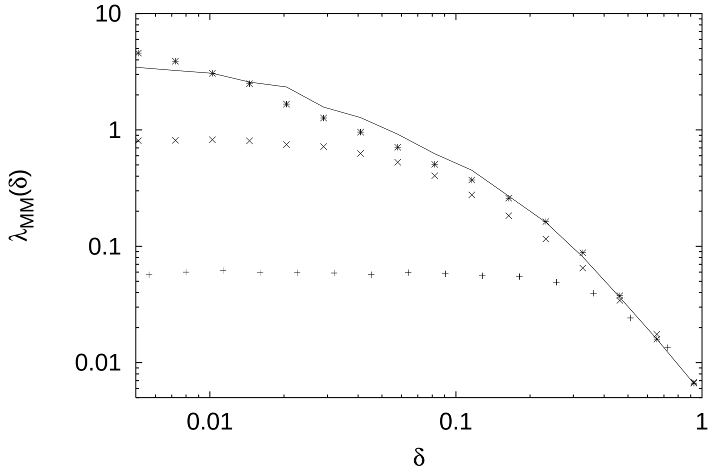

Figure 3. The FSLE for the eddy-viscosity shell model (18) $\lambda_{MM}(\delta)$ at various resolutions $N_M = 9(+), 15(\times), 20(*)$. For comparison the FSLE $\lambda_{TT}(\delta)$ curve is plotted (continuous curve). Here $\kappa = 0.4$, $k_0 = 0.05$.

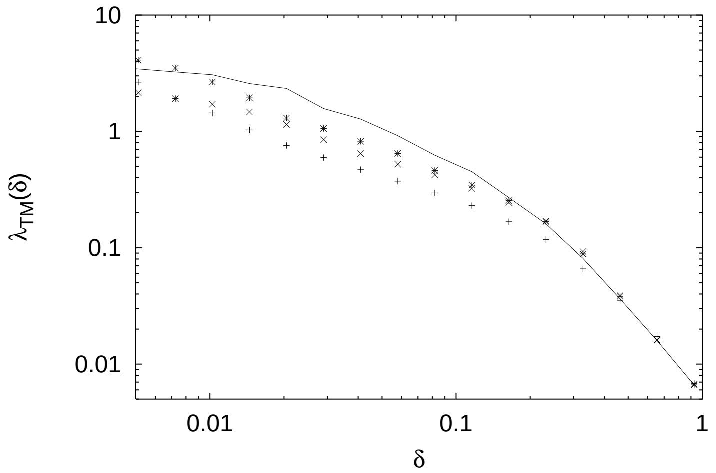

Figure 4. The FSLE between the eddy-viscosity shell model and the full shell model $\lambda_{TM}(\delta)$, at various resolutions $N_M = 9(+), 15(\times), 20(*)$. For comparison the FSLE $\lambda_{TT}(\delta)$ curve is plotted (continuous curve). The total number of shells for the complete model is $N = 24$, with $k_0 = 0.05$, $\nu = 10^{-7}$.

5. Conclusions

The lack of knowledge of the exact evolution equation for a real system is a widespread condition in scientific investigation. Even in those cases when the evolution law is known, it often happens that the large number of variables involved calls for some modelization in order to perform the numerical analysis.

In this paper the effects of uncertainties in the evolution laws on the predictability properties are investigated and quantitatively characterized by means of the FSLE. In particular, we have considered systems involving several characteristic scales and times.

In situations where there is a feedback on the small scales by the large ones, the dynamics of the former cannot be thoroughly discarded without affecting the ability to forecast large-scales features. In the general case, there is no systematic procedure to construct a good model for the small unresolved scales, i.e. a model such that the (second kind) predictability of large scales is no worse than the intrinsic (first kind) predictability of the true system. By means of the FSLE analysis we have shown that with a suitable parametrization of the small scales, e.g. the eddy-viscosity modelization in turbulence, the ability to predict the large scales is basically the same as one has in the case of uncertainties of the same size on the initial conditions.

Acknowledgments

We thank L Biferale for useful suggestions and discussions. This work was partially supported by INFM (Progetto Ricerca Avanzata TURBO) and by European Network *Intermittency in Turbulent Systems* (contract number FMRX-CT98-0175). A special acknowledgment goes to B Marani for warm and continuous support.

Appendix. Parametrization of small scales

Typically, a realistic problem (e.g. turbulence) involves many interacting degrees of freedom with different characteristic times. Let us indicate with $\boldsymbol{z}$ the state of the system under consideration, with an evolution law

$$\frac{\mathrm{d}\boldsymbol{z}}{\mathrm{d}t} = \boldsymbol{F}(\boldsymbol{z}) \qquad \boldsymbol{F}, \boldsymbol{z} \in \mathcal{R}^N. \tag{A1}$$

The dynamical variables $\boldsymbol{z}$ can be split into two sets

$$\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y}) \tag{A2}$$

where $\boldsymbol{x} \in \mathcal{R}^n$ and $\boldsymbol{y} \in \mathcal{R}^m$ ($N = n + m$), with $\boldsymbol{x}$ and $\boldsymbol{y}$ being respectively, the 'slow' and 'fast' variables. The distinction between slow and fast variables is often largely arbitrary.

The evolution equation (A1) is divided into two blocks, the first one containing the dynamics of the slow variables and the second one associated with the dynamics of the fast variables:

$$\begin{aligned} \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} &= \boldsymbol{F}_1(\boldsymbol{x}) + \boldsymbol{F}_2(\boldsymbol{x}, \boldsymbol{y}) \\ \frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t} &= \tilde{\boldsymbol{F}}_1(\boldsymbol{x}, \boldsymbol{y}) + \tilde{\boldsymbol{F}}_2(\boldsymbol{y}). \end{aligned} \tag{A3}$$

If one is only interested in the slow variables it is necessary to write an 'effective' equation for $\boldsymbol{x}$. As far as we know there is only one case for which it is simple to find the effective equations for $\boldsymbol{x}$. If the characteristic times of the fast variables are much smaller than those of the $\boldsymbol{x}$ (adiabatic limit), one can write

$$\boldsymbol{y} = \langle \boldsymbol{y} \rangle + \boldsymbol{\eta}(t) \tag{A4}$$

where $\boldsymbol{\eta}$ is a Wiener process, i.e. a zero-mean Gaussian process with

$$\langle \eta_i(t) \eta_j(t^{'}) \rangle = \langle \delta y_i^2 \rangle \delta_{ij} \delta(t - t^{'}). \tag{A5}$$

Therefore, for the slow variables one obtains

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{F}_1(\boldsymbol{x}) + \delta \boldsymbol{F}_1(\boldsymbol{x}) + \delta \boldsymbol{W}(\boldsymbol{x}, \boldsymbol{\eta}) \tag{A6}$$

where $\delta \boldsymbol{F}_1(\boldsymbol{x}) = \boldsymbol{F}_2(\boldsymbol{x}, \langle \boldsymbol{y} \rangle) + \delta \boldsymbol{F}_2$, $\delta F_{2,j} = \frac{1}{2} \sum_i \partial^2 F_{2,j} / \partial y_j^2 \langle \delta y_i^2 \rangle$ and $\delta W_i = \sum_i \partial F_{2,j} / \partial y_j |_{\langle y \rangle} \eta_i(t)$. Basically, the slow variables $\boldsymbol{x}$ obey a nonlinear Langevin equation.

Here the role of the fast degrees of freedom becomes relatively simple: they give small changes to the drift $\boldsymbol{F}_1 \to \boldsymbol{F}_1 + \delta \boldsymbol{F}_1$, and a noise term $\delta \boldsymbol{W}(\boldsymbol{x}, \boldsymbol{\eta})$. We remark that the validity of the above argument is rather limited. Even if one has a large timescale separation, the statistics of the fast variables can be very far from the Gaussian distribution. In particular, in a system with feedback ($\tilde{\boldsymbol{F}}_1 \neq 0$) one cannot model the fast variable $\boldsymbol{y}$ independently of the resolved $\boldsymbol{x}$.

In the generic situation the construction of the effective equation for $\boldsymbol{x}$ requires one to follow phenomenological arguments which depend on the physical mechanism of the particular problem. For example, for the Lorenz '96 model discussed in section 3, where $F_{2,k}(\boldsymbol{x}, \boldsymbol{y}) = \sum_{j=1,J} y_{j,k}$, we use the following procedure for the parametrization of the fast variables and the building of the effective equation for $\boldsymbol{x}$. Instead of assuming (A4) we mimic the fast variables in terms of the slow ones:

$$\boldsymbol{y}(t) = \boldsymbol{g}(\boldsymbol{x}(t)) = \langle \boldsymbol{y} | \boldsymbol{x}(t) \rangle + \boldsymbol{\eta}(t) \tag{A7}$$

where $\langle | \boldsymbol{x} \rangle$ stands for the conditional average and $\boldsymbol{\eta}(t)$ is a noise term. Inserting (A7) into the first equation of (A3) one obtains

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{F}_1(\boldsymbol{x}) + \boldsymbol{F}_2(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{F}_1(\boldsymbol{x}) + \boldsymbol{F}_2(\boldsymbol{x}, \langle \boldsymbol{y} | \boldsymbol{x} \rangle) + \delta \boldsymbol{F}_2(\boldsymbol{x}) \tag{A8}$$

where

$$\delta F_{2,i} = \sum_{j,k} \frac{\partial^2 F_{2,i}}{\partial y_j \partial y_k} \bigg|_{y = \langle y | x \rangle} \langle \eta_j \eta_k \rangle. \tag{A9}$$

In the Lorenz '96 model (10), because of the linear coupling between the different scales, the terms $\delta \boldsymbol{F}_2$ are absent and one has a close model for the large-scale variables:

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{F}_1(\boldsymbol{x}) + \boldsymbol{F}_2(\boldsymbol{x}, \langle \boldsymbol{y} | \boldsymbol{x} \rangle). \tag{A10}$$

The ansatz (A7) is well verified in the numerical simulations. We have computed the $\lambda_{TM}(\delta)$ by using a best fit for $\boldsymbol{F}_2$ and we have obtained a good reproduction of the $\lambda_{TT}(\delta)$ for large δ. In the Lorenz '96 model (10), where the coupling between slow and fast variables is practically linear, one has that $F_{2,k}(\boldsymbol{x}, \langle \boldsymbol{y} | \boldsymbol{x} \rangle) = \sum_{j=1,J} \langle y_{j,k} | x_k \rangle \simeq \nu_e x_k$.

Now we will discuss the case of the shell model parametrization which pertains to the general issue of the sub-grid modelization. The literature on this field and the related problems (e.g. closure in fully developed turbulence) is enormous and we do not pretend to discuss this field in detail here. Let us only recall the basic idea introduced over a century ago by Boussinesq, and later developed further by Taylor, Prandtl and Heisenberg—to mention some of the most notable names—for fully developed turbulence [25]. In a nutshell, the idea is to mimic the energy flux from the large to the small scales (in our terms from slow to fast variables) by an effective dissipation: the effect of the small scales on the large ones can be modelled as an enhanced molecular viscosity.

By simple dimensional arguments one can argue that the effects of small scales can be replaced by an effective viscosity at scales r, given by

$$\nu^{(e)} \sim r \delta v(r) \tag{A11}$$

where $\delta v(r)$ is the velocity fluctuation on the scale r.

The above argument for the shell model (13) gives [24]

$$\nu_n^{(e)} = \kappa \frac{|u_n|}{k_n} \tag{A12}$$

where $\kappa \sim O(1)$ is an empirical constant. From (A11) one could naively consider using a dimensional argument *á la* Kolmogorov to set a constant eddy viscosity $\nu_n^{(e)} \sim k_n^{-4/3}$. In this way one forgets the dynamics and this can cause numerical blow up. More sophisticated arguments that do not include the dynamics lead to similar problems.

Let us remark that the parametrization (A12) is not exactly identical to those obtained by closure approaches where the eddy viscosity is given in terms of averaged quantities. In our case this would mean writing $\langle u_n^2 \rangle^{1/2}$ instead of $|u_n|$ in (A12).

After this discussion it is easy to recognize that the parametrization in terms of conditional averages introduced for the Lorenz '96 model is, *a posteriori*, an eddy-viscosity model.

References

[1] Monin A 1973 *Weather Prediction as a Problem in Physics* (Cambridge, MA: MIT Press)

[2] Guckenheimer J and Holmes P 1983 *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields* (Berlin: Springer)

[3] Berkooz G 1994 An observation probability density equations, or, when simulations reproduce statistics? *Nonlinearity* **7** 313–28

[4] Holmes P J, Lumley J L and Berkooz G 1996 *Turbulence, Coherent Structures, Dynamical Systems and Symmetry* (Cambridge: Cambridge University Press) ch XII

[5] Crisanti A, Falcioni M and Vulpiani A 1989 On the effects of an uncertainty on the evolution law in dynamical systems *Physica* A **160** 482–502

[6] Benzi R, Paladin G, Parisi G and Vulpiani A 1985 Characterization of intermittency in chaotic systems *J. Phys. A: Math. Gen.* **18** 2157–65

[7] Paladin G and Vulpiani A 1987 Anomalous scaling in multifractal objects *Phys. Rep.* **156** 147–225

[8] Crisanti A, Jensen M H, Paladin G and Vulpiani A 1993 Intermittency and predictability in turbulence *Phys. Rev. Lett.* **70** 166–9

[9] Leith C E and Kraichnan R H 1972 Predictability of turbulent flows *J. Atmos. Sci.* **29** 1041–58

[10] Lorenz E N 1996 Predictability—a problem partly solved *Proc. Seminar on Predictability (Reading, UK) European Centre for Medium-Range Weather Forecast* pp 1–18

[11] Aurell E, Boffetta G, Crisanti A, Paladin G and Vulpiani A 1996 Growth of non-infinitesimal perturbations in turbulence *Phys. Rev. Lett.* **77** 1262–5

[12] Aurell E, Boffetta G, Crisanti A, Paladin G and Vulpiani A 1997 Predictability in the large: An extension of the concept of Lyapunov exponent *J. Phys. A: Math. Gen.* **30** 1–26

[13] Knuth D E 1969 *The Art of Computer Programming* vol 2 (Reading, MA: Addison-Wesley)

[14] Ruelle D 1979 Microscopic fluctuations and turbulence *Phys. Lett.* A **72** 81–2

[15] Lorenz E N 1963 Deterministic non-periodic flow *J. Atmos. Sci.* **20** 130–41

[16] Benettin G, Galgani L, Giorgilli A and Strelcyn J M 1980 Lyapunov characteristic exponent for smooth dynamical systems and Hamiltonian systems; a method for computing all of them *Meccanica* **15** 9–20

[17] Artale V, Boffetta G, Celani A, Cencini M and Vulpiani A 1997 Dispersion of passive tracers in closed basins: beyond the diffusion coefficient *Phys. Fluids* **9** 3162–71

[18] Boffetta G, Paladin G and Vulpiani A 1996 Strong chaos without butterfly effect in dynamical systems with feedback *J. Phys. A: Math. Gen.* **29** 2291–8

[19] Lorenz E N and Emanuel K A 1998 Optimal sites for supplementary weather observations: simulation with a small model *J. Atmos. Sci.* **55** 399–414

[20] Boffetta G, Giuliani P, Paladin G and Vulpiani A 1998 An extension of the Lyapunov analysis for the predictability problem *J. Atmos. Sci.* **55** 3409–16

[21] Bohr T, Jensen M H, Paladin G and Vulpiani A 1998 *Dynamical Systems Approach to Turbulence* (Cambridge: Cambridge University Press)

[22] L'vov V S, Podivilov E, Pomyalov A, Procaccia I and Vandembroucq D 1998 Improved shell model of turbulence *Phys. Rev.* E **58** 1811–22

[23] Lorenz E N 1969 The predictability of a flow which possesses many scales of motion *Tellus* **21** 289–307

[24] Benzi R, Biferale L, Succi S and Toschi F 1998 Intermittency and eddy-viscosities in dynamical models of turbulence *Phys. Fluids* **11** 1221–8

[25] Frisch U 1995 *Turbulence: the Legacy of A. N. Kolmogorov* (Cambridge: Cambridge University Press)